A RIGID BODY DYNAMICS DERIVED FROM A CLASS OF EXTENDED GAUDIN MODELS : AN INTEGRABLE DISCRETIZATION

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Abstract. We consider a hierarchy of classical Liouville completely integrable models sharing the same (linear) r–matrix structure obtained through an N–th jet–extension of $\mathfrak{su}(2)$ rational Gaudin models. The main goal of the present paper is the study of the integrable model corresponding to $N = 3$, since the case $N = 2$ has been considered by the authors in separate papers, both in the one–body case (Lagrange top) and in the n–body one (Lagrange chain). We now obtain a rigid body associated with a Lie–Poisson algebra which is an extension of the Lie–Poisson structure for the two–field top, thus breaking its semidirect product structure. In the second part of the paper we construct an integrable discretization of a suitable continuous Hamiltonian flow for the system. The map is constructed following the theory of Bäcklund transformations for finite–dimensional integrable systems developed by V.B. Kuznetsov and E.K. Sklyanin.

1. Introduction

In [1] we have considered Liouville completely integrable Hamiltonian systems with N degrees of freedom obtained through the N–th jet–extension of Gaudin models.

Such procedure allows one to construct a hierarchy of integrable models sharing the same (linear) r–matrix structure, whose first elements are, in the $\mathfrak{su}(2)$ case [1, 2] :

- $N = 1$: an Euler top associated with the Lie–Poisson algebra $\mathfrak{su}(2)$ [3, 4, 5].
- $N = 2$: a Lagrange top associated with the Lie–Poisson algebra $\mathfrak{e}(3) = \mathfrak{su}(2) \oplus_s \mathbb{R}^3$ [3, 4, 5].

Furthermore, a direct sum procedure allows to build long–range chains of n interacting bodies, with rational, trigonometric and elliptic r–matrices [1, 8]. For instance:

- $N = 1$: a Gaudin model associated with $\bigoplus_{i=1}^{n} \mathfrak{su}_i(2)$ [6, 7].
- $N = 2$: a Lagrange chain associated with $\bigoplus_{i=1}^{n} \mathfrak{e}_i(3)$ [1, 8].

Moreover this construction can be generalized to any finite–dimensional simple Lie algebra instead of $\mathfrak{su}(2)$ [1].

The aim of the present paper is to investigate the one–body system corresponding to $N = 3$, whose underlying algebra still includes $\mathfrak{su}(2)$ and $\mathbb{R}^3$ as proper subalgebras but is no longer a semidirect sum of subalgebras. This fact suggests one to see the $N = 3$ case as a slightly generalized version of the Lagrange top. The corresponding n-boby system will be considered in a separate paper.

We have here to mention that, up to our knowledge, the system considered in this paper has been introduced, in a different framework, by J.L. Thiffeault and P.J. Morrison in [9] and it is called the twisted Lagrange top. They study this model in the spirit of the dynamical systems theory, so that they do not use a Lax pair and an r–matrix approach, as we do in the present work. In [9] they obtain this new kind of integrable top adding a cocycle to the Lie–Poisson structure for the two–field top [5], thus breaking its semidirect product structure. We remark that in [9] the so–called twisted top remains a mathematical construction without a physical interpretation.

Later on, O. Vivolo, in [10], constructs a Lax matrix for such system, called here generalized Lagrange top. The integrability is proven by direct inspection since an r–matrix approach is not used, and the author focuses his attention on the study of the spectral curve of the system through the algebraic–geometry machinery. The main goal in [10] is the proof that the generalized Lagrange top has monodromy, as well as the standard Lagrange top, so that it does not admit global action–angle variables.

The main purpose of the present paper is the study of these generalized Lagrange tops using the r–matrix structure inherited from Gaudin models [1]. We obtain complete integrability for a large hierarchy of integrable systems by providing a Lax pair and a linear r–matrix algebra. In the second part of the paper we obtain an integrable discretization of a suitable continuous Hamiltonian flow for the system. The map

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is constructed following the theory of Bäcklund transformations for finite–dimensional integrable systems developed by V.B. Kuznetsov and E.K. Sklyanin in the papers [11, 12].

We would like here to remark that our approach allows one to consider also a generic finite–dimensional simple Lie algebra instead of $\mathfrak{su}(2)$ and a natural generalization to a n–body system with rational, trigonometric and elliptic dependences on the spectral parameter [1].

2. Jet-extensions of $\mathfrak{su}(2)$ rational Gaudin models

Let us consider one–body rational Lax matrices of the following form:

$$\mathcal{L}(\lambda) = \sum_{\alpha=1}^{3} \sigma^\alpha \left[b^\alpha + \sum_{i=0}^{N-1} f_i(\lambda)\, y_i^\alpha \right], \qquad f_0(\lambda) \doteq \frac{1}{\lambda}, \tag{2.0.1}$$

where $b^\alpha \in \mathbb{R}$, $\alpha = 1, 2, 3$ and

$$\sigma^1 \doteq \frac{1}{2} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^2 \doteq \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^3 \doteq \frac{1}{2} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

The $3N$ coordinate functions y_i^α, $\alpha = 1, 2, 3$, satisfy the following Lie–Poisson algebra:

$$\mathfrak{g}^{(N)} : \left\{ y_i^\alpha, y_j^\beta \right\} = \begin{cases} \epsilon^{\alpha\beta}{}_\gamma\, y_{i+j}^\gamma & i+j < N, \\ 0 & i+j \geq N. \end{cases} \tag{2.0.2}$$

Here $\epsilon^{\alpha\beta}{}_\gamma$ is the skew–symmetric tensor with $\epsilon^{12}{}_3 = 1$. Let us notice that N is exactly the order of the jet–extension of the Lie–Poisson algebra $\mathfrak{su}(2)$ [1]. It coincides with the number of degrees of freedom of the system.

The functions $f_i(\lambda)$ are chosen as

$$f_i(\lambda) = \sum_{\{\mathbf{q}_i\}} \prod_{k=1}^{i} \frac{c_k^{q_k}}{q_k!} \left(\frac{d}{d\lambda} \right)^{|\mathbf{q}_i|} f_0(\lambda), \qquad i = 1, \ldots, N-1,$$

where the c_k's are arbitrary complex constants and

$$\{\mathbf{q}_i\} \doteq \{\mathbf{q} \in \mathbb{N}^i \ : \ q_1 + 2\, q_2 + \cdots + i\, q_i = i\}, \qquad |\mathbf{q}_i| \doteq \sum_{k=1}^{i} q_k, \qquad i = 1, \ldots, N-1.$$

In [1] we have shown that the following proposition holds.

Proposition 1. *The Lax matrix (2.0.1) satisfies the linear r–matrix algebra*

$$\{\mathcal{L}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}(\mu)\} = [r(\lambda - \mu), \mathcal{L}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}(\mu)]\,, \tag{2.0.3}$$

$$r(\lambda) = \frac{1}{\lambda} \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha,$$

whit respect to the Lie–Poisson algebra (2.0.2). Here $\mathbb{1}$ is the 2×2 identity matrix.

The Lax matrix (2.0.1) can be written in an equivalent form as a 2×2 matrix with elements in the negative part of the loop–algebra $\mathfrak{g}^{(N)}[\lambda, \lambda^{-1}]$:

$$\mathcal{L}(\lambda) = \frac{\mathrm{i}}{2} \begin{pmatrix} u(\lambda) & v(\lambda) \\ w(\lambda) & -u(\lambda) \end{pmatrix}, \tag{2.0.4}$$

where

$$u(\lambda) = b^3 + \sum_{i=0}^{N-1} f_i(\lambda)\, y_i^3, \quad v(\lambda) = b^1 - \mathrm{i}\, b^2 + \sum_{i=0}^{N-1} f_i(\lambda)\, \left(y_i^1 - \mathrm{i}\, y_i^2 \right), \quad w(\lambda) = b^1 + \mathrm{i}\, b^2 + \sum_{i=0}^{N-1} f_i(\lambda)\, \left(y_i^1 + \mathrm{i}\, y_i^2 \right).$$

It is easy to see that equation (2.0.3) is equivalent to the following Lie-Poisson brackets for the rational functions $u(\lambda)$, $v(\lambda)$, $w(\lambda)$:

$$\begin{aligned} \{u(\lambda),u(\mu)\} &= \{v(\lambda),u(\mu)\} = \{w(\lambda),w(\mu)\} = 0, \\ \{u(\lambda),v(\mu)\} &= \frac{\mathrm{i}}{\lambda-\mu}[v(\lambda)-v(\mu)], \\ \{u(\lambda),w(\mu)\} &= \frac{\mathrm{i}}{\lambda-\mu}[w(\lambda)-w(\mu)], \\ \{v(\lambda),w(\mu)\} &= \frac{2\,\mathrm{i}}{\lambda-\mu}[u(\lambda)-u(\mu)]. \end{aligned}$$

Recall that the correspondence

$$(\xi^1,\xi^2,\xi^3)^T \in \mathbb{R}^3 \;\longmapsto\; \xi = \frac{1}{2}\begin{pmatrix} \mathrm{i}\,\xi^3 & \mathrm{i}\,\xi^1+\xi^2 \\ \mathrm{i}\,\xi^1-\xi^2 & -\,\mathrm{i}\,\xi^3 \end{pmatrix} \in \mathfrak{su}(2),$$

is an isomorphism between $\mathfrak{su}(2)$ and the Lie algebra $(\mathbb{R}^3,[\cdot,\cdot])$, where the Lie bracket $[\cdot,\cdot]$ is realized with the wedge product $\wedge$. Note that

$$\langle \xi,\eta\rangle = -2\,\mathrm{tr}\,(\xi\,\eta) = 2\,\mathrm{tr}\,(\eta\,\xi^*),$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^3$.

We now focus our attention on an interesting reduction of the Lax matrix (2.0.1). If we assume $c_1=-1$ and $c_k=0$ for $2\le k\le N-1$ we readily get

$$\mathcal{L}(\lambda) = \sum_{\alpha=1}^{3}\sigma^\alpha\left[b^\alpha + \sum_{i=0}^{N-1}\frac{y_i^\alpha}{\lambda^{i+1}}\right]. \tag{2.0.5}$$

Let us now fix the following notation:

$$\mathbf{y}_i \doteq (y_i^1,y_i^2,y_i^3)^T \in \mathbb{R}^3 \quad i=0,...,N-1, \qquad \mathbf{b} \doteq (b^1,b^2,b^3)^T \in \mathbb{R}^3.$$

The complete integrability of the hierarchy of systems obtained from the Lax matrices (2.0.5) corresponding to different values of N is established by the following statements.

Lemma 1. *The characteristic curve $\Gamma^{(N)}:\ \det(\mathcal{L}(\lambda)-\mu\,\mathbb{1})=0$ is an hyperelliptic curve of the following form:*

$$\Gamma^{(N)}:\ 4\,\mu^2 + \langle\mathbf{b},\mathbf{b}\rangle + H(\lambda) + C(\lambda) = 0, \tag{2.0.6}$$

where

$$H(\lambda) = \sum_{i=1}^{N}\frac{1}{\lambda^i}\left[2\,\langle\mathbf{b},\mathbf{y}_{i-1}\rangle + \sum_{k=1}^{i-2}\langle\mathbf{y}_k,\mathbf{y}_{i-k-2}\rangle\right], \tag{2.0.7}$$

$$C(\lambda) = \sum_{i=N+1}^{2N}\frac{1}{\lambda^i}\sum_{k=i-N-1}^{N-1}\langle\mathbf{y}_k,\mathbf{y}_{i-k-2}\rangle. \tag{2.0.8}$$

Proof: A straightforward computation.

□

Proposition 2. *The curve (2.0.6) provides a set of $2N$ Poisson–commuting integrals of motion given by*

$$H_k = 2\,\langle\mathbf{b},\mathbf{y}_{k-1}\rangle + \sum_{i=0}^{k-2}\langle\mathbf{y}_i,\mathbf{y}_{k-i-2}\rangle, \qquad C_k = \sum_{i=k-1}^{N-1}\langle\mathbf{y}_i,\mathbf{y}_{N+k-i-2}\rangle, \qquad k=1,...,N,$$

$$\{H_i,H_k\} = \{C_i,H_k\} = \{C_i,C_k\} = 0, \qquad i,k=1,...,N.$$

The integrals H_k, $k=1,...,N$ are first integrals of motion. The integrals C_k, $k=1,...,N$ are linear combinations of the Casimir functions of the Lie–Poisson algebra $\mathfrak{g}^{(N)}$, namely

$$\{C_k,y_j^\beta\}=0 \quad \forall\, y_j^\beta\in\mathfrak{g}^{(N)},\ \forall\, k=1,...,N.$$

Proof: The quantities H_k and C_k, $k=1,...,N$, are immediately obtained through the following formulae:

$$H_k = \mathrm{Res}_{\lambda=0}\, \lambda^{k-1} H(\lambda), \qquad C_k = \mathrm{Res}_{\lambda=0}\, \lambda^{N+k-1} C(\lambda), \qquad k=1,...,N,$$

where $H(\lambda)$ and $C(\lambda)$ are given respectively in (2.0.7) and (2.0.8). The function $H(\lambda)$ provides N Poisson–commuting first integrals of motion thanks to the r–matrix structure (2.0.3). The fact that $C(\lambda)$ is a generating function for the Casimirs of $\mathfrak{g}^{(N)}$ can be proven by a direct computation:

$$\begin{aligned}\{C_k, y_j^\beta\} &= \sum_{\alpha=1}^{3}\sum_{i=k-1}^{N-1} \{y_i^\alpha\, y_{N+k-i-2}^\alpha, y_j^\beta\} = \sum_{\alpha=1}^{3}\sum_{i=k-1}^{N-1}\left[y_i^\alpha \{y_{N+k-i-2}^\alpha, y_j^\beta\} + y_{N+k-i-2}^\alpha \{y_i^\alpha, y_j^\beta\}\right] = \\ &= 2\sum_{\alpha=1}^{3}\sum_{i=k-1}^{N-1} \epsilon^{\alpha\beta}{}_{\gamma}\, y_{i+j}^\gamma\, y_{N+k-i-2+j}^\alpha,\end{aligned}$$

for all $y_j^\beta \in \mathfrak{g}^{(N)}$, $k=1,...,N$. Now, if $i+j \geq N$ then $\{C_k, y_j^\beta\}$=0 thanks to (2.0.2). Let us consider $i+j<N$:

$$\{C_k, y_j^\beta\} = \sum_{\alpha=1}^{3} \epsilon^{\alpha\beta}{}_{\gamma}\left[\sum_{i=k-1}^{N-1} y_{i+j}^\gamma\, y_{N+k-i-2+j}^\alpha + \sum_{i'=k-1}^{N-1} y_{i'+j}^\alpha\, y_{N+k-i'-2+j}^\gamma\right] = 0,$$

where $i' = N+k-i-2$.

□

3. The 3-rd jet extension of $\mathfrak{su}(2)$ rational Gaudin models

The Lax matrix

$$\mathcal{L}(\lambda) = \sum_{\alpha=1}^{3} \sigma^\alpha \left[b^\alpha + \frac{y_0^\alpha}{\lambda} - \frac{c_1\, y_1^\alpha}{\lambda^2} + \frac{y_2^\alpha}{\lambda^2}\left(\frac{c_1^2}{\lambda} - c_2\right)\right], \tag{3.0.9}$$

is obtained considering $N=3$ in (2.0.1). Here $b^\alpha \in \mathbb{R}$, $\alpha=1,2,3$ plays the role of an external field, taken as uniform and constant (in time), c_1, c_2 are real arbitrary constants and the 9 coordinate functions y_i^α, $\alpha=1,2,3$, $i=0,1,2$ satisfy the Lie–Poisson brackets (2.0.2) with $N=3$ (i.e. $\mathfrak{g}^{(3)}$), namely:

$$\begin{gathered}\left\{y_0^\alpha, y_0^\beta\right\} = \epsilon^{\alpha\beta}{}_{\gamma}\, y_0^\gamma, \qquad \left\{y_0^\alpha, y_1^\beta\right\} = \epsilon^{\alpha\beta}{}_{\gamma}\, y_1^\gamma, \qquad \left\{y_0^\alpha, y_2^\beta\right\} = \epsilon^{\alpha\beta}{}_{\gamma}\, y_2^\gamma, \\ \left\{y_1^\alpha, y_1^\beta\right\} = \epsilon^{\alpha\beta}{}_{\gamma}\, y_2^\gamma, \qquad \left\{y_1^\alpha, y_2^\beta\right\} = 0, \qquad \left\{y_2^\alpha, y_2^\beta\right\} = 0.\end{gathered} \tag{3.0.10}$$

Let us notice that

$$\mathfrak{g}^{(3)} = \mathfrak{su}(2) \oplus_s \mathcal{G}, \qquad \dim \mathcal{G} = 6,$$

where $\mathcal{G}$, although including the abelian proper subalgebra $\mathbb{R}^3$ spanned by $\mathbf{y}_2$, doesn't have a semidirect structure. In [9] it is shown that our algebra $\mathfrak{g}^{(3)}$ can be obtained adding a cocycle to $\mathfrak{e}(3,2) = \mathfrak{su}(2) \oplus_s (\mathbb{R}^3 \oplus \mathbb{R}^3)$. We recall that it is possible to use the Lie–Poisson algebra $\mathfrak{e}(3,2)$ to describe a two–field top [5].

The Lie–Poisson bracket between two functions $f, g : \mathfrak{g}^{(3)} \to \mathbb{R}$ reads

$$\begin{aligned}\{f(\mathbf{y}_0,\mathbf{y}_1,\mathbf{y}_2), g(\mathbf{y}_0,\mathbf{y}_1,\mathbf{y}_2)\} &= \langle \mathbf{y}_0, \nabla_{\mathbf{y}_0} f \wedge \nabla_{\mathbf{y}_0} g\rangle + \langle \mathbf{y}_1, \nabla_{\mathbf{y}_0} f \wedge \nabla_{\mathbf{y}_1} g + \nabla_{\mathbf{y}_1} f \wedge \nabla_{\mathbf{y}_0} g\rangle + \\ &+ \langle \mathbf{y}_2, \nabla_{\mathbf{y}_0} f \wedge \nabla_{\mathbf{y}_2} g + \nabla_{\mathbf{y}_2} f \wedge \nabla_{\mathbf{y}_0} g\rangle + \langle \mathbf{y}_2, \nabla_{\mathbf{y}_1} f \wedge \nabla_{\mathbf{y}_1} g\rangle,\end{aligned}$$

where ∇ is a gradient with respect to its subscript. The non semidirect structure lies just in the last term of the above equation: this term is absent in the $\mathfrak{e}(3,2)$ algebra.

The Lie–Poisson algebra $\mathfrak{g}^{(3)}$ has 3 Casimir functions:

$$C^{(1)} = \langle \mathbf{y}_0, \mathbf{y}_2\rangle + \frac{1}{2}\langle \mathbf{y}_1, \mathbf{y}_1\rangle, \qquad C^{(2)} = \langle \mathbf{y}_1, \mathbf{y}_2\rangle, \qquad C^{(3)} = \langle \mathbf{y}_2, \mathbf{y}_2\rangle,$$

which differ from the $\mathfrak{e}(3,2)$ Casimirs just for the presence of $\langle \mathbf{y}_0, \mathbf{y}_2\rangle$ in $C^{(1)}$.

Specializing Lemma 1 to the $N=3$ case we immediately have the statement

Proposition 3. *The characteristic curve* $\Gamma^{(3)} : \det(\mathcal{L}(\lambda) - \mu\,\mathbb{1}) = 0$ *provides a set of 6 Poisson–commuting integrals of motion:*

$$\Gamma^{(3)} :\ 4\,\mu^2 + \langle \mathbf{b}, \mathbf{b}\rangle + \frac{H_1}{\lambda} + \frac{H_2}{\lambda^2} + \frac{H_3}{\lambda^3} + \frac{C_1}{\lambda^4} + \frac{C_2}{\lambda^5} + \frac{C_3}{\lambda^6} = 0, \tag{3.0.11}$$

where

$$\begin{aligned}
H_1 &= 2\,\langle \mathbf{b}, \mathbf{y}_0\rangle, \\
H_2 &= \langle \mathbf{y}_0, \mathbf{y}_0\rangle - 2\,c_1\,\langle \mathbf{b}, \mathbf{y}_1 + c_2\,\mathbf{y}_2\rangle, \\
H_3 &= 2\,c_1\left(c_1\,\langle \mathbf{b}, \mathbf{y}_2\rangle - \langle \mathbf{y}_0, \mathbf{y}_1 + c_2\,\mathbf{y}_2\rangle\right), \\
C_1 &= c_1^2\left(c_2^2\,C^{(3)} + 2\,c_2\,C^{(2)} + 2\,C^{(1)}\right), \\
C_2 &= -2\,c_1^3\left(c_2\,C^{(3)} + C^{(2)}\right), \\
C_3 &= c_1^4\,C^{(3)}.
\end{aligned}$$

4. A special reduction of the 3-rd jet extension

Let us now consider the Lax matrix (2.0.5) with $N = 3$ and $\mathbf{b} = (0, 0, b) \in \mathbb{R}^3$. To simplify the notations we prefer to rename the 9 coordinate functions y_i^α, $\alpha = 1, 2, 3$, $i = 0, 1, 2$ as $y_0^\alpha \doteq y^\alpha$, $y_1^\alpha \doteq x^\alpha$, $y_2^\alpha \doteq z^\alpha$, $\alpha = 1, 2, 3$. Thus we obtain the following Lax matrix:

$$\mathcal{L}(\lambda) = b\,\sigma^3 + \sum_{\alpha=1}^{3} \sigma^\alpha \left[\frac{y^\alpha}{\lambda} + \frac{x^\alpha}{\lambda^2} + \frac{z^\alpha}{\lambda^3}\right]. \tag{4.0.12}$$

Notice that (4.0.12) is the extension to a third–order pole (i.e. three degrees of freedom) of the Lagrange top Lax matrix [2, 5]. Thus, we may expect a generalization of the Lagrange system.

First, let us notice that the characteristic curve $\Gamma^{(3)} : \det(\mathcal{L}(\lambda) - \mu\,\mathbb{1}) = 0$ (3.0.11) provides the following integrals of motion:

$$H_1 = 2\,\langle \mathbf{b}, \mathbf{y}\rangle, \qquad H_2 = \langle \mathbf{y}, \mathbf{y}\rangle + 2\,\langle \mathbf{b}, \mathbf{x}\rangle, \qquad H_3 = 2\,\left(\langle \mathbf{b}, \mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle\right), \tag{4.0.13}$$

$$C_1 = 2\,C^{(1)} = 2\,\langle \mathbf{y}, \mathbf{z}\rangle + \langle \mathbf{x}, \mathbf{x}\rangle, \qquad C_2 = 2\,C^{(2)} = 2\,\langle \mathbf{x}, \mathbf{z}\rangle, \qquad C_3 = C^{(3)} = \langle \mathbf{z}, \mathbf{z}\rangle. \tag{4.0.14}$$

As in the Lagrange case the third component of $\mathbf{y}$ and the euclidean norm of $\mathbf{z}$ are constants of the motion. Looking at the brackets (3.0.10), and taking into account that $\mathbf{y}$ and $\mathbf{z}$ span respectively $\mathfrak{su}(2)$ and $\mathbb{R}^3$, we may interpret them as the total angular momentum of the system and the vector pointing from a fixed point (which we shall take as $(0, 0, 0) \in \mathbb{R}^3$) to the centre of mass of an axially symmetric rigid body, namely a Lagrange top. Let us remark that $\mathbf{y}$ does not coincide with the angular momentum of the top due to the presence of the vector $\mathbf{x}$. We think of $\mathbf{x}$, whose norm in not constant, as the position of the moving centre of mass of the global system composed by the Lagrange top and a material point, whose position is described by $\mathbf{x} - \mathbf{z}$. Here we are assuming that both bodies have unitary masses. The link between these two systems is given by integrals C_1 and C_2. If we think of a canonical realization of the Lie–Poisson algebra $\mathfrak{g}^{(3)}$ (3.0.10) in term of three canonical coordinates and their conjugated momenta we can immediately argue that the vector $\mathbf{x}$ must depend on momenta, since $\left\{x^\alpha, x^\beta\right\} = \epsilon^{\alpha\beta}{}_\gamma\, z^\gamma$. We come back to this point in more detail in section 5.

If we look at the first integral

$$H_2 = \langle \mathbf{y}, \mathbf{y}\rangle + 2\,\langle \mathbf{b}, \mathbf{x}\rangle,$$

we immediately see that it coincides with the physical Hamiltonian of the Lagrange top, where the vector $\mathbf{y}$ is the angular momentum of the spinning top and the vector $\mathbf{x}$ describes the motion of the centre of mass of the the top on the surface $|\mathbf{x}| = c$, with c constant. In the case of the Lagrange top the equations of motion (in the rest frame) with respect to the Hamiltonian H_2 can be written in the following form:

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{b} \wedge \mathbf{x}, \\ \dot{\mathbf{x}} = \mathbf{y} \wedge \mathbf{x}, \end{cases}$$

which indicate that $\mathbf{x}$ rotates rigidly. In the following subsection we shall derive the equations of motions for the system described by (4.0.12) emphasizing the main differences with the Lagrange case.

4.1. **A Lax representation.** First, it is useful to introduce the following complex generators:

$$y^{\pm} = y^1 \pm \mathrm{i}\, y^2, \qquad x^{\pm} = x^1 \pm \mathrm{i}\, x^2, \qquad z^{\pm} = z^1 \pm \mathrm{i}\, z^2.$$

In term of $(y^3, y^{\pm}), (x^3, x^{\pm}), (z^3, z^{\pm})$ the Lie–Poisson algebra $\mathfrak{g}^{(3)}$ reads

$$\begin{gathered}
\{y^3, y^{\pm}\} = \mp \mathrm{i}\, y^{\pm}, \qquad \{y^+, y^-\} = -2\,\mathrm{i}\, y^3, \\
\{y^3, x^{\pm}\} = \{x^3, y^{\pm}\} = \mp \mathrm{i}\, x^{\pm}, \quad \{y^+, x^-\} = \{x^+, y^-\} = -2\,\mathrm{i}\, x^3, \quad \{y^3, x^3\} = \{y^+, x^+\} = \{y^-, x^-\} = 0, \\
\{y^3, z^{\pm}\} = \{z^3, y^{\pm}\} = \mp \mathrm{i}\, z^{\pm}, \quad \{y^+, z^-\} = \{z^+, y^-\} = -2\,\mathrm{i}\, z^3, \quad \{y^3, z^3\} = \{y^+, z^+\} = \{y^-, z^-\} = 0, \\
\{x^3, x^{\pm}\} = \mp \mathrm{i}\, z^{\pm}, \qquad \{x^+, x^-\} = -2\,\mathrm{i}\, z^3, \\
\{x^{\alpha}, z^{\beta}\} = \{z^{\alpha}, z^{\beta}\} = 0, \qquad \alpha, \beta = \pm, 3.
\end{gathered}$$

The Lax matrix (4.0.12) has the following simple form:

$$\mathcal{L}(\lambda) = \mathcal{B} + \frac{\mathcal{Y}}{\lambda} + \frac{\mathcal{X}}{\lambda^2} + \frac{\mathcal{Z}}{\lambda^3}, \tag{4.1.1}$$

where

$$\mathcal{B} \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad \mathcal{Y} \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} y^3 & y^- \\ y^+ & -y^3 \end{pmatrix}, \quad \mathcal{X} \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} x^3 & x^- \\ x^+ & -x^3 \end{pmatrix}, \quad \mathcal{Z} \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} z^3 & z^- \\ z^+ & -z^3 \end{pmatrix},$$

are all $\mathfrak{su}(2)$ matrices.

We choose as a physical Hamiltonian of the system the first integral H_2:

$$\mathcal{H} = \frac{1}{2} H_2 = \frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle + \langle \mathbf{b}, \mathbf{x}\rangle = -\mathrm{Tr}\,\left(\mathcal{Y}^2 + 2\,\mathcal{B}\,\mathcal{X}\right). \tag{4.1.2}$$

We see that the kinetic term is given by the norm of the total angular momentum (up to a factor 1/2), while the potential energy is given by the projection of the total centre of mass vector onto the external field.

In the case of the Hamiltonian (4.1.2) the equations of motion are given by

$$\left\{ \begin{array}{l} \dot{\mathbf{y}} = \mathbf{b} \wedge \mathbf{x}, \\ \dot{\mathbf{x}} = \mathbf{y} \wedge \mathbf{x} + \mathbf{b} \wedge \mathbf{z}, \\ \dot{\mathbf{z}} = \mathbf{y} \wedge \mathbf{z}, \end{array} \right. \qquad \Leftrightarrow \qquad \left\{ \begin{array}{l} \dot{\mathcal{Y}} = [\mathcal{B}, \mathcal{X}], \\ \dot{\mathcal{X}} = [\mathcal{Y}, \mathcal{X}] + [\mathcal{B}, \mathcal{Z}], \\ \dot{\mathcal{Z}} = [\mathcal{Y}, \mathcal{Z}]. \end{array} \right. \tag{4.1.3}$$

We immediately see that the vector $\mathbf{x}$ does not rotate rigidly, though $\mathbf{z}$ still does. Obviously, since $|\mathbf{x}|$ is no longer preserved, the integral $C_2 = 2\,C^{(2)} = 2\,\langle \mathbf{x}, \mathbf{z}\rangle$ does not imply that the angle between $\mathbf{x}$ and $\mathbf{z}$ is constant. Using equations (4.1.3) we obtain that the evolution equation for the vector pointing from $(0,0,0) \in \mathbb{R}^3$ to the position of the material point is given by:

$$\frac{d}{dt}(\mathbf{x} - \mathbf{z}) = \mathbf{y} \wedge (\mathbf{x} - \mathbf{z}) + \mathbf{b} \wedge \mathbf{z}$$

We conclude this section giving a Lax representation for equations of motion (4.1.3). The proof is straightforward.

Proposition 4. *The Lax representation for equations (4.1.3) is given by*

$$\dot{\mathcal{L}}(\lambda) \doteq \frac{d}{dt}\mathcal{L}(\lambda) = \left[\mathcal{L}(\lambda), \mathcal{M}(\lambda)\right], \qquad \mathcal{M}(\lambda) = \frac{\mathcal{X}}{\lambda} + \frac{\mathcal{Z}}{\lambda^2}.$$

5. A canonical realization of the reduced system

As we have shown in the previous section, our model is a Hamiltonian system. Our aim is now to find a coordinate transformation on the symplectic leaves that makes the system canonical. We will use three Euler angles $\theta \in [0, 2\pi)$, $\phi \in [0, 2\pi)$ and $\psi \in [0, \pi)$ with their canonical conjugate momenta p_θ, p_ϕ amd p_ψ.

Our canonical description is restricted to the following symplectic leaf:

$$\mathcal{O} \doteq \left\{ (\mathbf{y}, \mathbf{x}, \mathbf{z}) \in \mathfrak{g}^{(3)} |\; C^{(1)} = 0,\; C^{(2)} = 0,\; C^{(3)} = 1 \right\}. \tag{5.0.4}$$

Proposition 5. *A canonical realization of the Lie–Poisson algebra $\mathfrak{g}^{(3)}$ restricted to the symplectic leaf $\mathcal{O}$ (5.0.4) is given by:*

$$\begin{aligned}
\mathbf{y} &= \left(\sin\phi\, p_\theta + \cot\theta\, \cos\phi\, p_\phi - \frac{\cos\phi}{\sin\theta}\, p_\psi, -\cos\phi\, p_\theta + \cot\theta\, \sin\phi\, p_\phi - \frac{\sin\phi}{\sin\theta}\, p_\psi, p_\phi \right), \\
\mathbf{x} &= \sqrt{2\, p_\psi}\, \left(\sin\psi\, \sin\phi - \cos\theta\, \cos\psi\, \cos\phi, -\sin\psi\, \cos\phi - \cos\theta\, \cos\psi\, \sin\phi, -\sin\theta\, \cos\psi \right), \\
\mathbf{z} &= \left(\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta \right).
\end{aligned}$$

Proof: It is well–known [13] that the canonical realization of an angular momentum $\mathbf{y}$ and of a constant vector $\mathbf{z}$ such that $|\mathbf{z}| = 1$ in the euclidean space is given by

$$\mathbf{y} = \left(\sin\phi\, p_\theta + \cot\theta\,\cos\phi\, p_\phi - \frac{\cos\phi}{\sin\theta}\, p_\psi, -\cos\phi\, p_\theta + \cot\theta\,\sin\phi\, p_\phi - \frac{\sin\phi}{\sin\theta}\, p_\psi, p_\phi\right),$$

$$\mathbf{z} = (\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta),$$

where $q \doteq (\theta, \phi, \psi)$ are the standard Euler angles and $p \doteq (p_\theta, p_\phi, p_\psi)$ are their canonical conjugated momenta. Let us recall that if $f(q,p)$ and $g(q,p)$ are two arbitrary functions then

$$\{f(q,p), g(q,p)\} = \sum_{i=1}^{3}\left[\frac{\partial f(q,p)}{\partial q_i}\frac{\partial g(q,p)}{\partial p_i} - \frac{\partial f(q,p)}{\partial p_i}\frac{\partial g(q,p)}{\partial q_i}\right].$$

Requiring the Lie–Poisson brackets (3.0.10) restricted to the symplectic leaf $\mathcal{O}$ (5.0.4) we easily obtain

$$\mathbf{x} = \sqrt{2\, p_\psi}\,(\sin\psi\,\sin\phi - \cos\theta\,\cos\psi\,\cos\phi, -\sin\psi\,\cos\phi - \cos\theta\,\cos\psi\,\sin\phi, -\sin\theta\,\cos\psi).$$

□

As we mentioned in section 3 we have finally obtained the vector $\mathbf{x}$ described in term of canonical coordinates θ, ϕ, ψ and the conjugated momentum p_ψ. In particular, we have $|\mathbf{x}|^2 = 2\, p_\psi > 0$. As a consequence the Hamiltonian $\mathcal{H}$ (4.1.2) describes a non–holonomic dynamics.

For the sake of completeness we write the first integrals of motion using the above canonical description. The physical Hamiltonian (4.1.2) takes the following form:

$$\mathcal{H} = \frac{p_\theta^2}{2} + \frac{p_\psi^2 + p_\phi^2 - 2\, p_\psi\, p_\phi\,\cos\theta}{2\sin^2\theta} - b\,\sqrt{2\, p_\psi}\,\sin\theta\cos\psi.$$

We see that the variable ϕ is cyclic as in the Lagrange case, while ψ explicitly enters in the potential term. The remaining first integrals of motion are:

$$I_1 = \frac{1}{2}H_2 = p_\phi,$$
$$I_2 = \frac{1}{2}H_3 = \sqrt{2\, p_\psi}\,[p_\theta\,\sin\psi + (p_\psi - p_\phi\,\cos\theta)\cot\theta\,\cos\psi - p_\phi\,\sin\theta\,\cos\psi] + b\,\cos\theta.$$

6. An integrable discretization through Bäcklund transformations

The theory of integrable maps got a boost when Veselov developed a theory of Lagrange correspondences [14, 15]. These maps are symplectic multi–valued transformations which have enough integrals of motion, this definition being a proper analog of the classical Liouville integrability. In the main examples, studied by him and later by other authors, the integrable maps are constructed as time–discretizations of classical integrable models, see, for instance, [14, 15, 4]. Moreover these correspondences associate to a given solution of an integrable system a new solution, a property reminescent of Bäcklund transformations (BTs) for soliton equations.

In this paper we apply the theory of BT for finite–dimensional integrable systems, developed by V.B. Kuznetsov, E.K. Sklyanin and P. Vanhaecke in the relevant papers [11, 12]. Following this approach we look at BTs as special Poisson maps. It is possible to find an exhaustive list of the features of these BTs in [11, 12]. Some of these are the following ones:

(1) a BT is a Poisson map that preserves the same set of integrals of motion as does the continuous flow which it discretizes;
(2) a BT is given by explicit formulae rather then implicit equations;
(3) although a BT is multi–valued, it leads to a single–valued map on any level manifold of the integrals of motion;
(4) when one searches for the simplest BT of an integrable system, then one finds a one–dimensional family $\{\mathcal{B}_\eta | \eta \in \mathbb{C}\}$ of them. The Bäcklund parameter η is canonically conjugate to μ, i.e. $\mu = -\partial F/\partial\eta$ with F_η generating function of $\{\mathcal{B}_\eta | \eta \in \mathbb{C}\}$. Here μ is bound to η by the equation of an algebraic curve (dependent on the integrals), which is exactly the characteristic curve that appears in the linearization of the integrable system. This property is called spectrality of the BT;
(5) the explicit nature of a BT makes it purely iterative, so that it is very well suited as symplectic integrator for the underlying model. Here the parameter η is an adjustable discrete time step.

6.1. **One–point BT.** First, let us consider the Lax matrix of our reduced system in the form (2.0.4), where the entries of $\mathcal{L}(\lambda)$ are given by

$$u(\lambda) = b + \frac{y^3}{\lambda} + \frac{x^3}{\lambda^2} + \frac{z^3}{\lambda^3}, \qquad v(\lambda) = \frac{y^-}{\lambda} + \frac{x^-}{\lambda^2} + \frac{z^-}{\lambda^3}, \qquad w(\lambda) = \frac{y^+}{\lambda} + \frac{x^+}{\lambda^2} + \frac{z^+}{\lambda^3}.$$

A one–point BT can be defined as the following similarity transform on the Lax matrix $\mathcal{L}(\lambda)$:

$$\mathcal{B}_\eta : \; \mathcal{L}(\lambda) \longmapsto \mathcal{M}(\lambda;\eta)\, \mathcal{L}(\lambda)\, \mathcal{M}^{-1}(\lambda,\eta) \qquad \forall\, \lambda \in \mathbb{C}, \quad \eta \in \mathbb{C},$$

with some generically non-degenerate 2×2 matrix $\mathcal{M}(\lambda,\eta)$, simply because a BT should preserve the spectrum of $\mathcal{L}(\lambda)$. The parameter η is called Bäcklund parameter.

We use $\tilde{}$-notations for the updated variables, so that

$$\mathcal{B}_\eta : \; \mathcal{L}(\lambda) \longmapsto \; \tilde{\mathcal{L}}(\lambda) = \frac{\mathrm{i}}{2} \begin{pmatrix} \tilde{u}(\lambda) & \tilde{v}(\lambda) \\ \tilde{w}(\lambda) & -\tilde{u}(\lambda) \end{pmatrix},$$

$$\tilde{u}(\lambda) = b + \frac{\tilde{y}^3}{\lambda} + \frac{\tilde{x}^3}{\lambda^2} + \frac{\tilde{z}^3}{\lambda^3}, \qquad \tilde{v}(\lambda) = \frac{\tilde{y}^-}{\lambda} + \frac{\tilde{x}^-}{\lambda^2} + \frac{\tilde{z}^-}{\lambda^3}, \qquad \tilde{w}(\lambda) = \frac{\tilde{y}^+}{\lambda} + \frac{\tilde{x}^+}{\lambda^2} + \frac{\tilde{z}^+}{\lambda^3}.$$

We shall consider the similarity transformation between $\mathcal{L}(\lambda)$ and $\tilde{\mathcal{L}}(\lambda)$, namely

$$\mathcal{M}(\lambda;\eta)\, \mathcal{L}(\lambda) = \tilde{\mathcal{L}}(\lambda)\, \mathcal{M}(\lambda;\eta) \qquad \forall\, \lambda \in \mathbb{C}, \quad \eta \in \mathbb{C}, \tag{6.1.1}$$

with the following intertwining matrix [2, 16]:

$$\mathcal{M}(\lambda;\eta) = \begin{pmatrix} \lambda - \eta + p\, q & p \\ q & 1 \end{pmatrix}, \qquad \det \mathcal{M}(\lambda;\eta) = \lambda - \eta. \tag{6.1.2}$$

Note that the number of zeros of $\det \mathcal{M}$ is the number of essential Bäcklund parameters. Moreover the variables p and q are indeterminate dynamical variables.

Comparing the asymptotics in $\lambda \to \infty$ in both sides of (6.1.1) we readily get

$$p = \frac{y^-}{2\,b}, \qquad q = \frac{\tilde{y}^+}{2\,b}, \qquad \tilde{y}^3 = y^3. \tag{6.1.3}$$

If we want an explicit map from $\mathcal{L}(\lambda)$ to $\tilde{\mathcal{L}}(\lambda)$ we must express q in term of the old variables. To solve this problem one can use the spectrality of the BTs [11, 12]. Equation (6.1.1) defines a map $\mathcal{B}_P$ parametrized by the point $P = (\eta, \mu) \in \Gamma^{(3)}$. Notice that there are two points on $\Gamma^{(3)}$, $P = (\eta,\mu)$ and $Q = (\eta, -\mu)$, corresponding to the same η and sitting one above the other because of the hyperelliptic involution:

$$(\eta, \mu) \in \Gamma^{(3)} : \qquad \det(\mathcal{L}(\eta) - \mu\, \mathbb{1}) = 0.$$

This spectrality property give us the formula [2, 16]

$$q = \frac{u(\eta) - \mu}{v(\eta)} = -\frac{w(\eta)}{u(\eta) + \mu}, \tag{6.1.4}$$

where η and μ are bounded by the algebraic curve

$$-\mu^2 = \frac{1}{4}\left(\langle \mathbf{b}, \mathbf{b} \rangle + \frac{H_1}{\eta} + \frac{H_2}{\eta^2} + \frac{H_3}{\eta^3} + \frac{C_1}{\eta^4} + \frac{C_2}{\eta^5} + \frac{C_3}{\eta^6} \right),$$

and the integrals H_i and C_i, $i = 1, 2, 3$, are given respectively in (4.0.13) and (4.0.14). Now the equation (6.1.1) gives an integrable Poisson map from $\mathcal{L}(\lambda)$ to $\tilde{\mathcal{L}}(\lambda)$:

$$\begin{aligned}
\tilde{u}(\lambda) &= \frac{(\lambda - \eta + 2\, p\, q)[u(\lambda) - q\, v(\lambda)] + p\, w(\lambda)}{\lambda - \eta}, \\
\tilde{v}(\lambda) &= \frac{(\lambda - \eta + 2\, p\, q)^2 v(\lambda) - 2\, p\, (\lambda - \eta + 2\, p\, q) u(\lambda) - p^2 w(\lambda)}{\lambda - \eta}, \\
\tilde{w}(\lambda) &= \frac{w(\lambda) + 2\, q\, u(\lambda) - q^2 v(\lambda)}{\lambda - \eta}.
\end{aligned}$$

Collecting the negative powers of λ the above formulae can be rewritten as an explicit map

$$\mathcal{B}_\eta : \; (\mathbf{y}, \mathbf{x}, \mathbf{z}) \; \longmapsto \; (\tilde{\mathbf{y}}, \tilde{\mathbf{x}}, \tilde{\mathbf{z}}),$$

given by

$$
\begin{array}{rcl}
\tilde{y}^3 &=& y^3, \\
\tilde{y}^- &=& x^- + (p\,q - \eta)\,y^- - 2\,p\,y^3, \\
\tilde{y}^+ &=& 2\,q\,b, \\
\tilde{x}^3 &=& x^3 + p\,y^+ - q\,x^- - q\,(pq\,-\eta)\,y^- + 2\,p\,q\,y^3, \\
\tilde{x}^- &=& (2\,p\,q - \eta)\,x^- - 2\,p\,x^{\,3} - p^2\,y^+ + p\,q\,(p\,q - \eta)\,y^- - 2\,p^2\,q\,y^3, \\
\tilde{x}^+ &=& y^+ - \frac{q}{p}(p\,q - \eta)\,y^- + 2\,q\,y^3, \\
\tilde{z}^3 &=& z^3 - q\,z^- + 2\,p\,q\,x^3 - p\,q^2\,x^- + p\,x^+ + \eta\,q\,(p\,q - \eta)\,y^- + p\,\eta\,y^+ + 2\,\eta\,p\,q\,y^3, \\
\tilde{z}^+ &=& 2\,q\,x^3 - q^2\,x^- + x^+ + \eta\,\frac{q}{p}(p\,q - \eta)\,y^- + \eta\,y^+ + 2\,\eta\,q\,y^3, \\
\tilde{z}^- &=& (2\,p\,q - \eta)\,z^- - 2\,p\,z^3 - 2\,p^2\,q\,x^3 + p^2\,q^2\,x^- - p^2\,x^+ - \eta\,p\,q(p\,q - \eta)y^- - \\
&& -p^2\,\eta\,y^+ - 2\,\eta\,p^2\,q\,y^3.
\end{array} \tag{6.1.5}
$$

The following statement shows how the one–point BT can be written in a symplectic form through a generating function. We restrict our BT $\mathcal{B}_\eta$ to a symplectic leaf of the Lie–Poisson structure by fixing values of the Casimir functions $C^{(1)}, C^{(2)}, C^{(3)}$:

$$
\mathcal{O} \doteq \left\{ (\mathbf{y}, \mathbf{x}, \mathbf{z}) \in \mathfrak{g}^{(3)} |\; C^{(1)} = \gamma_1,\; C^{(2)} = \gamma_2,\; C^{(3)} = 1 \right\}.
$$

Let us fix the following notation:

$$
\Psi \doteq (y^3, x^3, z^3)^T, \qquad \tilde{\Psi} \doteq (\tilde{y}^3, \tilde{x}^3, \tilde{z}^3)^T,
$$

$$
\chi^- \doteq (y^-, x^-, z^-)^T, \qquad \tilde{\chi}^+ \doteq (\tilde{y}^+, \tilde{x}^+, \tilde{z}^+)^T.
$$

Proposition 6. *The one–point BT $\mathcal{B}_\eta|_{\mathcal{O}}$ can be arranged in the form:*

$$
\Psi_i = \sum_{j=1}^{3} \{\Psi_i, \chi_j^-\}\, \nabla^j_{\chi^-} F_\eta(\chi^- | \tilde{\chi}^+), \tag{6.1.6}
$$

$$
\tilde{\Psi}_i = \sum_{j=1}^{3} \{\tilde{\chi}_j^+, \tilde{\Psi}_i\}\, \nabla^j_{\tilde{\chi}^+} F_\eta(\chi^- | \tilde{\chi}^+), \tag{6.1.7}
$$

$i = 1, 2, 3$, where

$$
\begin{array}{rcl}
F_\eta(\chi^- | \tilde{\chi}^+) &=& \dfrac{y^-\,\tilde{y}^+}{2\,b} + k\left(\dfrac{y^-}{z^-} + \dfrac{\tilde{y}^+}{\tilde{z}^+}\right) - \dfrac{(1+\eta\,\gamma_2)^2}{4\,k\,\eta^2} + \dfrac{1}{2}\left(\dfrac{\gamma_2^2}{4} - \gamma_1\right)\ln\left(\dfrac{k+1}{k-1}\right) - \\
&& -\dfrac{1}{2\,k}\left[\tilde{z}^+\,x^- + z^-\,\tilde{x}^+ - \eta\,\tilde{x}^+\,x^- + \dfrac{x^-}{z^-}\left(\dfrac{x^-}{z^-} + \dfrac{\eta}{2}x^-\,\tilde{z}^+ - \gamma_2\right) + \dfrac{\tilde{x}^+}{\tilde{z}^+}\left(\dfrac{\tilde{x}^+}{\tilde{z}^+} + \dfrac{\eta}{2}\tilde{x}^+\,z^- - \gamma_2\right)\right],
\end{array} \tag{6.1.8}
$$

with

$$
k^2 = 1 + \eta\,z^-\,\tilde{z}^+.
$$

Proof: The Casimir functions $C^{(1)}, C^{(2)}, C^{(3)}$ do not change under the map:

$$
\mathcal{B}_\eta:\; (C^{(1)}, C^{(2)}, C^{(3)}) \longmapsto (\tilde{C}^{(1)}, \tilde{C}^{(2)}, \tilde{C}^{(3)}) = (C^{(1)}, C^{(2)}, C^{(3)}).
$$

The above invariance allows one to exclude 6 variables, expressing y^+, x^+, z^+ and $\tilde{y}^-, \tilde{x}^-, \tilde{z}^-$ in term of the components of the vectors Ψ, χ^- and $\tilde{\Psi}, \tilde{\chi}^+$:

$$
\begin{aligned}
y^+ &= \frac{1}{z^-}\left[2\,\gamma_1 - 1 - 2\,y^3\,z^3 - \frac{y^-}{z^-}(1-(z^3)^2)\right],\\
x^+ &= \frac{1}{z^-}\left[2\,\gamma_2 - 2\,x^3\,z^3 - \frac{x^-}{z^-}(1-(z^3)^2)\right],\\
z^+ &= \frac{1-(z^3)^2}{z^-},\\
\tilde{y}^- &= \frac{1}{\tilde{z}^+}\left[2\,\gamma_1 - 1 - 2\,\tilde{y}^3\,\tilde{z}^3 - \frac{\tilde{y}^+}{\tilde{z}^+}(1-(\tilde{z}^3)^2)\right],\\
\tilde{x}^- &= \frac{1}{\tilde{z}^+}\left[2\,\gamma_2 - 2\,\tilde{x}^3\,\tilde{z}^3 - \frac{\tilde{x}^+}{\tilde{z}^+}(1-(\tilde{z}^3)^2)\right],\\
\tilde{z}^- &= \frac{1-(\tilde{z}^3)^2}{\tilde{z}^+}.
\end{aligned}
$$

With the help of (6.1.3) we can rewrite equations (6.1.5) of the map in the following form:

$$
\begin{aligned}
\Psi_1 = y^3 &= \frac{y^-\,y^+}{2\,b} - \frac{1}{2\,k^3}\left\{\frac{1}{2}\left[\eta\left(z^-\,\tilde{x}^+ + x^-\,\tilde{z}^+\right) - z^-\,\tilde{z}^+\right]^2 + \eta^2\,\left[(z^-)^2\tilde{z}^+\,\tilde{y}^+ + (\tilde{z}^+)^2 z^-\,y^-\right] + \right.\\
&\quad + \left(1+\frac{1}{2}\eta\,\gamma_2\right)\left(z^-\,\tilde{x}^+ + x^-\,\tilde{z}^+\right) - (\gamma_2 + 2\,\eta\,\gamma_1)\,z^-\,\tilde{z}^+ - \eta\,\left(x^-\,\tilde{x}^+ + z^-\,\tilde{y}^+ + \tilde{z}^+\,y^-\right) + \\
&\quad \left. + \frac{1}{4}\left(\gamma_2^2 - 4\,\gamma_1\right)\right\},\\
\Psi_2 = x^3 &= \frac{1}{2\,k}\left[\tilde{y}^+\,x^- + \eta\,\left(x^-\,\tilde{z}^+ + \tilde{x}^+\,z^-\right) - \tilde{z}^+\,z^- + \gamma_2\right],\\
\Psi_3 = z^3 &= \frac{\tilde{y}^+\,z^-}{2\,w} + k,\\
\tilde{\Psi}_1 = \tilde{y}^3 &= y^3,\\
\tilde{\Psi}_2 = \tilde{x}^3 &= \frac{1}{2\,k}\left[\tilde{x}^+\,y^- + \eta\,\left(x^-\,\tilde{z}^+ + \tilde{x}^+\,z^-\right) - \tilde{z}^+\,z^- + \gamma_2\right],\\
\tilde{\Psi}_3 = \tilde{z}^3 &= \frac{\tilde{z}^+\,y^-}{2\,w} + k,
\end{aligned}
$$

where $k^2 = 1 + \eta\,z^-\,\tilde{z}^+$. It is now easy to check that the function $F_\eta(\chi^-|\tilde{\chi}^+)$ (6.1.8) satisfies equations (6.1.6), (6.1.7).

□

Remark 1. It is possible to use the theory of canonical transformations to show that $\mathcal{B}_\eta$ has the spectrality property. The spectrality property of a BT means that the two coordinates η and μ parametrizing the map are conjugated variables, namely

$$\mu = -\frac{\partial F_\eta}{\partial \eta},$$

where F_η is the generating function of the BT.

In our case, using equations (6.1.3), (6.1.4) and (6.1.8), we obtain

$$\mu = u(\eta) - \frac{\tilde{y}^+}{2\,b}\,v(\eta) = -\frac{\partial F_\eta(\chi^-|\tilde{\chi}^+)}{\partial \eta},$$

so that the spectrality property holds.

Remark 2. We have here to remark that the one–parameter map (6.1.5) is a complex transformation, namely it is not a physical BT for our system. Following [2] we shall construct, in the next subsection, a physical map using two Bäcklund parameters.

6.2. **Two–point BT.** According to [2, 16], we construct a composite map which is a product of the map $\mathcal{B}_{P_1} \doteq \mathcal{B}_{(\eta_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \doteq \mathcal{B}_{(\eta_2,-\mu_2)}$:

$$\mathcal{B}_{P_1,Q_2} = \mathcal{B}_{Q_2} \circ \mathcal{B}_{P_1} : \ \mathcal{L}(\lambda) \overset{\mathcal{B}_{P_1}}{\longmapsto} \tilde{\mathcal{L}}(\lambda) \overset{\mathcal{B}_{Q_2}}{\longmapsto} \breve{\mathcal{L}}(\lambda).$$

The two maps are inverse to each other when $\eta_1 = \eta_2$ and $\mu_1 = \mu_2$. This two-point BT is defined by the following discrete–time Lax equation:

$$\mathcal{M}(\lambda;\eta_1,\eta_2)\,\mathcal{L}(\lambda) = \breve{\mathcal{L}}(\lambda)\,\mathcal{M}(\lambda;\eta_1,\eta_2) \qquad \forall\lambda\in\mathbb{C}, \qquad \eta_1,\eta_2\in\mathbb{C}, \tag{6.2.1}$$

where the matrix $\mathcal{M}(\lambda;\eta_1,\eta_2)$ is [2, 16]

$$\mathcal{M}(\lambda;\eta_1,\eta_2) = \begin{pmatrix} \lambda-\eta_1+s\,t & t \\ -s^2\,t+(\eta_1-\eta_2)\,s & \lambda-\eta_2-s\,t \end{pmatrix}, \tag{6.2.2}$$

$$\det\mathcal{M}(\lambda;\eta_1,\eta_2) = (\lambda-\eta_1)(\lambda-\eta_2).$$

The spectrality property with respect to two fixed points $(\eta_1,\mu_1)\in\Gamma^{(3)}$ and $(\eta_2,\mu_2)\in\Gamma^{(3)}$ give

$$s = \frac{u(\eta_1)-\mu_1}{v(\eta_1)} = \frac{\breve{u}(\eta_2)-\mu_2}{\breve{v}(\eta_2)}, \tag{6.2.3}$$

$$t = \frac{(\eta_1-\eta_2)(u(\eta_1)+\mu_1)(u(\eta_2)-\mu_2)}{(u(\eta_1)+\mu_1)w(\eta_2)-(u(\eta_2)-\mu_2)w(\eta_1)} = \frac{(\eta_1-\eta_2)(\breve{u}(\eta_1)-\mu_1)(\breve{u}(\eta_2)+\mu_2)}{(\breve{u}(\eta_2)+\mu_2)\breve{w}(\eta_1)-(\breve{u}(\eta_1)-\mu_1)\breve{w}(\eta_2)}. \tag{6.2.4}$$

Now we have two Bäcklund parameters $\eta_1,\eta_2\in\mathbb{C}$. It is possible to obtain several equivalent formulae [2, 16] for the variables s and t since the points (η_1,μ_1) and (η_2,μ_2) belong to the spectral curve $\Gamma^{(3)}$, i.e. are bound by the following relations

$$-(2\,\mu_j)^2 = u^2(\eta_j)+v(\eta_j)\,w(\eta_j) = \breve{u}^2(\eta_j)+\breve{v}(\eta_j)\,\breve{w}(\eta_j), \qquad j=1,2.$$

Together with (6.2.3) and (6.2.4), the formula (6.2.1) gives an explicit two–point Poisson integrable map from $\mathcal{L}(\lambda)$ to $\breve{\mathcal{L}}(\lambda)$. The map is parametrized by the two points P_1 and Q_2.

Obviously, when $\eta_1=\eta_2$ (and $\mu_1=\mu_2$) the map turns into an identity map. As we have shown in [2] the two–point BT can be reduced to a real Poisson integrable map if the following condition holds:

$$\eta_1 = \bar{\eta}_2 \doteq \eta = \Re(\eta)+\mathrm{i}\,\Im(\eta)\in\mathbb{C}.$$

Therefore, the two–point map leads to a physical BT $\mathcal{B}_\eta$ with two real parameters.

Let us now introduce the following notation

$$X \doteq (\mathbf{y},\mathbf{x},\mathbf{z})^T\in\mathbb{R}^9, \qquad \breve{X} \doteq (\breve{\mathbf{y}},\breve{\mathbf{x}},\breve{\mathbf{z}})^T\in\mathbb{R}^9.$$

A direct computation based on the similarity transform (6.2.1) shows that

Proposition 7. *The two–point BT $\mathcal{B}_{P_1,Q_2}|_{\eta_1=\bar{\eta}_2} : X \longmapsto \breve{X}$ is given by*

$$\breve{X} = \Phi(s,t;\eta)\,X + X_0(s,t;\eta), \tag{6.2.5}$$

with

$$\Phi(s,t;\eta) = \begin{pmatrix} \mathbb{1}_{3\times3} & \mathbb{0}_{3\times3} & \mathbb{0}_{3\times3} \\ A(s,t;\eta) & \mathbb{1}_{3\times3} & \mathbb{0}_{3\times3} \\ B(s,t;\eta) & A(s,t;\eta) & \mathbb{1}_{3\times3} \end{pmatrix},$$

where $A(s,t;\eta)$ and $B(s,t;\eta)$ are two 3×3 dynamical matrices depending on the Bäcklund parameter η and the parameters s,t and $X_0(s,t;\eta)$ is a dynamical vector. The matrices $\mathbb{1}_{3\times3}$ and $\mathbb{0}_{3\times3}$ are respectively the 3×3 identity matrix and the 3×3 zero matrix.

The explicit expressions of $X_0(s,t;\eta), A(s,t;\eta), B(s,t;\eta)$ are rather complicated and they are given in Appendix 1.

Remark 3. Notice that, despite its matrix formulation, the map (6.2.5) is a non linear transformation.

7. Concluding remarks

We have considered a hierarchy of classical Liouville completely integrable models sharing the same (linear) r–matrix structure obtained through an N–th jet–extension of $\mathfrak{su}(2)$ rational Gaudin models. The general procedure of such extension is presented in [1].

We have fixed $N = 3$ obtaining a rigid body associated to a Lie–Poisson algebra which is an extension of the Lie–Poisson structure for the two–field top. We have here to recall that this classical system has been introduced in [9] where it is called the twisted Lagrange top, and furtherly investigated in [10] in the algebraic–geometry setting.

The novelty of our approach is the introduction of an r–matrix formulation for this system. Its knowledge enables us to easily find a Lax formulation for the equation of motion, as well as a canonical realization in terms of Euler angles. Finally, through the approach developed by V.B. Kuznetsov, E.K. Sklyanin and P. Vanhaecke [11, 12], we find explicit BTs for the system. In Appendix 2 we present a numerical simulation of the real reduction of the two–point BT.

Let us remark that another feature of our approach is the natural possibility of constructing n–body integrable chains starting from each N–th jet–extension, both considering any simple Lie algebra and rational, trigonometric and elliptic dependances on the spectral parameter. This was done for the $N = 2$ case in [?]. The case $N = 3$ will be considered in a separate paper.

Appendix 1: Explicit expressions of $X_0(s,t;\eta), A(s,t;\eta), B(s,t;\eta)$

As we have shown in section 6.2 the two–point BT can be written in the following form:

$$\mathcal{B}_\eta:\; X \longmapsto \breve{X} = \Phi(s,t;\eta)\, X + X_0(s,t;\eta),$$

where

$$\Phi(s,t;\eta) = \begin{pmatrix} \mathbb{1}_{3\times3} & \mathbb{0}_{3\times3} & \mathbb{0}_{3\times3} \\ A(s,t;\eta) & \mathbb{1}_{3\times3} & \mathbb{0}_{3\times3} \\ B(s,t;\eta) & A(s,t;\eta) & \mathbb{1}_{3\times3} \end{pmatrix},$$

and $X_0(s,t;\eta)$ is a dynamical vector depending on η, s, t just as like the 3×3 matrices $A(s,t;\eta)$ and $B(s,t;\eta)$.

Here we recall that the dynamical variables s and t can be obtained using the spectrality property of the BT and are given by formulas (6.2.3) and (6.2.4), namely:

$$s = \frac{u(\eta_1) - \mu_1}{v(\eta_1)}, \qquad t = \frac{(\eta_1 - \eta_2)(u(\eta_1) + \mu_1)(u(\eta_2) - \mu_2)}{(u(\eta_1) + \mu_1)w(\eta_2) - (u(\eta_2) - \mu_2)w(\eta_1)}.$$

The explicit expressions of $X_0(s,t;\eta)$ is given by

$$\begin{aligned}
[X_0(s,t;\eta)]_1 &= b\,(s\,\alpha_1 - t),\\
[X_0(s,t;\eta)]_2 &= -b\,(s\,\alpha_1 + t),\\
[X_0(s,t;\eta)]_3 &= 0,\\
[X_0(s,t;\eta)]_4 &= b\,\left[\frac{s\,\alpha_1}{2}(\alpha_2 + \Re(\eta)) + t\,(\alpha_2 - 2\,\Re(\eta))\right],\\
[X_0(s,t;\eta)]_5 &= \mathrm{i}\,b\,\left[\frac{s\,\alpha_1}{2}(\alpha_2 + \Re(\eta)) - t\,(\alpha_2 - 2\,\Re(\eta))\right],\\
[X_0(s,t;\eta)]_6 &= 2\,b\,t\,s\,\alpha_1,\\
[X_0(s,t;\eta)]_7 &= -\frac{\mathrm{i}\,b}{4}\left[(\Im(\eta) + \Re(\eta))^2(t + \alpha_1) + 8\,s\,t\,\Re(\eta)(t - \alpha_1)\right],\\
[X_0(s,t;\eta)]_8 &= -\frac{b}{4}\left[(\Im(\eta) + \Re(\eta))^2(t - \alpha_1) + 8\,s\,t\,\Re(\eta)(t + \alpha_1)\right],\\
[X_0(s,t;\eta)]_9 &= 2\,[X_0(s,t;\eta)]_6\,\Re(\eta),
\end{aligned}$$

where we have introduced the quantities

$$\begin{aligned}
\alpha_1 &= 2\,\Im(\eta) - s\,t,\\
\alpha_2 &= \alpha_1 - s\,t,
\end{aligned}$$

and $\Im(\eta), \Re(\eta)$ denote respectively the imaginary part and the real part of the Bäcklund parameter η.

The entries of the dynamical matrix $A(s,t;\eta)$ are given by:

$$\begin{aligned}
&[A(s,t;\eta)]_{ii} = 0, \quad i = 1,2,3,\\
&[A(s,t;\eta)]_{12} = \mathrm{i}\,\alpha_2 = -[A(s,t;\eta)]_{21},\\
&[A(s,t;\eta)]_{13} = s\,\alpha_1 - t = -[A(s,t;\eta)]_{31},\\
&[A(s,t;\eta)]_{23} = -\mathrm{i}\,(s\,\alpha_1 + t) = -[A(s,t;\eta)]_{32},
\end{aligned}$$

so that $A(s,t;\eta)$ is a skew–symmetric matrix. The matrix $B(s,t;\eta)$ has a more complicated form: its entries are

$$\begin{aligned}
[B(s,t;\eta)]_{11} &= \frac{1}{2}(1-s^2)(\alpha_1^2 - t^2),\\
[B(s,t;\eta)]_{12} &= -\frac{\mathrm{i}}{2}[(t^2 - s^2\,\alpha_1^2) - 2\,\alpha_2\,\Re(\eta)],\\
[B(s,t;\eta)]_{13} &= \frac{1}{2}s\,\alpha_1\,(\alpha_2 + \Re(\eta)),\\
[B(s,t;\eta)]_{21} &= -\frac{\mathrm{i}}{2}[(t^2 - s^2\,\alpha_1^2) + 2\,\alpha_2\,\Re(\eta)],\\
[B(s,t;\eta)]_{22} &= \frac{1}{2}(1+s^2)(\alpha_1^2 - t^2)\\
[B(s,t;\eta)]_{23} &= -\frac{\mathrm{i}}{2}s\,\alpha_1\,(\alpha_2 + \Re(\eta)),\\
[B(s,t;\eta)]_{31} &= \frac{1}{2}\left[t\,(\alpha_2 + 2\,\Re(\eta)) + s\,\alpha_1\,(\alpha_2 - 2\,\Re(\eta))\right],\\
[B(s,t;\eta)]_{32} &= \frac{1}{2}\left[t\,(\alpha_2 + 2\,\Re(\eta)) - s\,\alpha_1\,(\alpha_2 - 2\,\Re(\eta))\right],\\
[B(s,t;\eta)]_{33} &= 4\,s\,t\,\alpha_1.
\end{aligned}$$

Appendix 2: Numerics

In this appendix we present a 3D plot corresponding to the real reduction of the two–point BT (6.2.5). It is obtained using a MAPLE 8 program that is a slightly different version of the MATLAB program developed by V.B. Kuznetsov in [2].

The input parameters are:

- the intensity of the external field, i.e. b;
- the Bäcklund parameter $\eta = \Re(\eta) + \mathrm{i}\,\Im(\eta)$. Here $\Im(\eta)$ is the time–step of the discretization;
- the number of iteration of the map, N;
- the initial values of the coordinate functions $y^1, y^2, y^3, x^1, x^2, x^3, z^1, z^2, z^3$.

The output is a 3D plot of $N+N$ consequent points $(x^1 - z^1, x^2 - z^2, x^3 - z^3)$ and (z^1, z^2, z^3). We remark that the vector $(x^1 - z^1, x^2 - z^2, x^3 - z^3)$ describes the position of the material point, as explained in section 4, and the vector (z^1, z^2, z^3) is the position of the centre of mass of the spinning top. As expected, the points (z^1, z^2, z^3) lie on the sphere $C^{(3)} = \langle \mathbf{z}, \mathbf{z} \rangle =$constant, of some radius defined by the initial data.

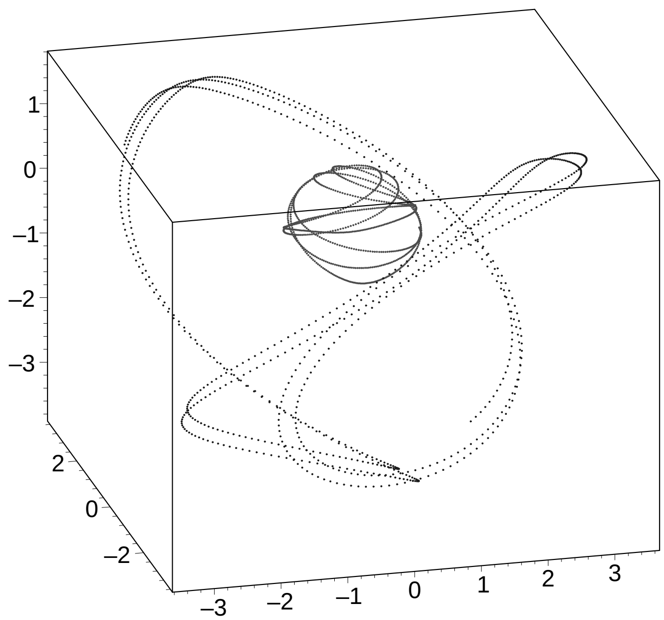

FIGURE 1. $(b;\Re(\eta),\Im(\eta);N;y^1,y^2,y^3,x^1,x^2,x^3,z^1,z^2,z^3)=(1;5,0.1;1000;-2.4,-0.6,-1.2,-2.19,0.89,1.34,1,0,0)$

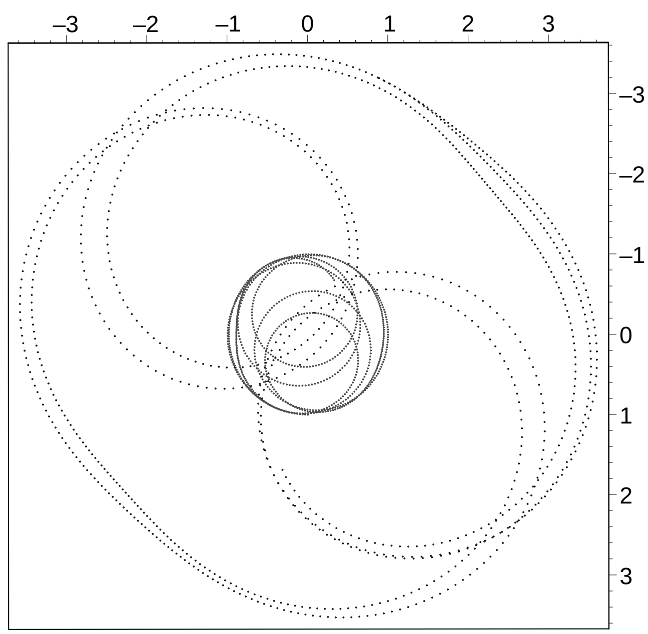

FIGURE 2. Projection of Figure 1 on the $x-y$ plane

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